0514 Rozansky

Review : category of MFs

joint work with M. Khovanov

discovered by D. Eisenbud in relation to hypersurface singularity

$W(x_1, \cdots, x_n) = 0$

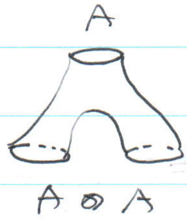
Landau - Ginzburg 2d TQFTs (Frobenius algebra)

Q_{BRST} $Q_{BRST}^2 = 0$ が成立つなら

$Q_{BRST}^2 = -W$ (bdry of 2d mfd)

Kontsevich : matrix fact. = bdry cond.

⇝ Kapustin and Li



$A^{\otimes 2} \longrightarrow A$

categorification にMFを使う.

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R: comm. algebra

Kom(free R-modules)

<u>objects</u> : (A, d) A: (free) $\mathbb{Z}$-graded R-module

$d \in \mathrm{End}_R(A)$ $\deg d = -1$, $d^2 = 0$

$$\xrightarrow{d} A_i \xrightarrow{d} A_{i-1} \rightarrow$$

<u>morphisms</u> : $A, B \in \mathrm{Kom}(R\text{-mod})$

$\mathrm{Hom}_R(A,B) \circlearrowleft \check{d}$ $\check{d}F = [\overset{d_A+d_B}{\overset{\shortparallel}{d}}, F]$

$= d_B F - (-1)^{|F|} F d_A$

$$d^{\vee 2}F = (d, (d, F)) \underset{\substack{\text{Jacobi}\\ \text{id.}}}{=} (d^2, F) = 0$$

$$\mathrm{Ext}(A,B) = \underset{\substack{\uparrow\\ \text{homs that commute with } d}}{\mathrm{Ker}\, d^\vee} / \underbrace{\mathrm{Im}\, d^\vee}_{\text{up to homotopy}}$$

$\mathrm{Hom}(A,B) = \mathrm{Ext}_0(A,B)$ degree 0 part

For $W \in R$

<u>Object</u> : $(A,D)_{W,R}$, where A- free $\mathbb{Z}_2$-graded R-module

$D \in \mathrm{End}_R(A)$ $\deg_{\mathbb{Z}_2} D = 1$

$D^2 = W \cdot \mathbb{1}$

$$\begin{tikzcd} A_1 \arrow[r, shift left, "P"] & A_0 \arrow[l, shift left, "Q"] \end{tikzcd} \qquad D = \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix}$$

$PQ = W \cdot 1_{A_0}$, $QP = W \cdot 1$

may write $A_1 \xrightarrow{P} A_0 \xrightarrow{Q} A_1$

If $W \neq 0$, then $\mathrm{rank}\, A_0 = \mathrm{rank}\, A_1$

If $W = 0$, define $H(A) = \mathrm{Ker}\, D / \mathrm{Im}\, D$

<u>morphisms</u> : $\mathrm{Hom}_R(A,B) \circlearrowleft \check{d}$ $\check{d}F = [D,F]$

$$\check{d}^2 F = [\underset{\substack{\parallel \\ W \cdot \mathbb{1}}}{D^2}, F] = 0$$

Define $\mathrm{Ext}(A,B) = \mathrm{Ker}\, \check{d} / \mathrm{Im}\, \check{d}$: $\mathbb{Z}_2$-graded

<u>Example</u>

$$\begin{tikzcd} R_1 \arrow[r, shift left, "1"] \arrow[d, "1"'] & R_0 \arrow[l, shift left, "W"] \arrow[d, "1"] \arrow[dl, dashed, "1"] \\ R_1 \arrow[r, shift left] & R_0 \arrow[l, shift left, "W"] \end{tikzcd}$$

$R_i = R$

contractible

MF_W triangulated

$$(A_1 \overset{P}{\underset{Q}{\rightleftarrows}} A_0)[1] = (A_0 \overset{-Q}{\underset{-P}{\rightleftarrows}} A)$$

If $F \in \mathrm{Ext}_1(A,B)$

$$\mathrm{Cone}\,F = \begin{tikzcd} A_1 \arrow[r, "P"] \arrow[ddr, "F_0"] & A_0 \arrow[r, "Q"] \arrow[ddr, "F_1"] & A_1 \\ \oplus & \oplus & \oplus \\ B_1 \arrow[r, "P'"'] & B_0 \arrow[r, "Q'"'] & B_1 \end{tikzcd}$$

$$D = D_A + D_B + F$$

$$D^2 = D_A^2 + D_B^2 + \underbrace{\{F, D\}}_{\overset{\shortparallel}{0}}$$

$$= W \cdot \mathbb{1}_A + W \cdot \mathbb{1}_B$$

$$= W \cdot \mathbb{1}_{A \oplus B}$$

<u>Example</u> Koszul complex

$$\begin{tikzcd} R_1 \arrow[r, "p"] \arrow[dr, "1"] & R_0 \\ R_1' \arrow[r, "q"] & R_0' \end{tikzcd}$$

$\vdots$

$p, q, r \in R$

cone を取る.

$$R_1 \xrightarrow{1} R_0' \oplus pR_0$$

base に同一視する

$$R_1' - qR_1' \xrightarrow[-pq]{} R_0$$

$W = pqr$

$$\begin{tikzcd} R_1 \arrow[r, "p"] \arrow[dr, "1"'] & R_0 \arrow[r, "qr"] \arrow[dr, "-r"] & R_1 \\ R_1' \arrow[r, "q"'] & R_0' \arrow[r, "pr"'] & R_1' \end{tikzcd}$$

$$\cong$$

$$R_1 \xrightarrow{1} R_0' + pR_0 \xrightarrow{pqr} R_1$$

$$R_1' - qR_1' \xrightarrow{-pq} R_0' \xrightarrow{-r} R_1' - qR_1$$

tensor products

$$(A, D_A)_{W_A, R} \otimes_R (B, D_B)_{W_B, R}$$

$$= (A \otimes_R B,\ D_A \otimes \mathbb{1} + (-1)^{\deg \mathbb{Z}_2} \otimes D_B)_{W_A + W_B, R}$$

$$(D_A \otimes \mathbb{1} + (-1)^{\deg \mathbb{Z}_2} \otimes D_B)^2 = D_A^2 + D_B^2 + \overbrace{[D_A, D_B]}^{\overset{0}{\shortparallel}}$$

$$= W_A \mathbb{1}_{A \otimes B} + W_B \cdot \mathbb{1}_{A \otimes B}$$

$A \in MF_{W,R}$, $B \in MF_{\underset{\sim}{-W},R}$

$$\mathrm{Tor}(A,B) = H(A \otimes B) \quad (\because D^2 = 0)$$

$$A \hat{\otimes}_R B = A \otimes_R B$$

forget R-mod str.
and consider it as a vector sp.
⋯ this is q.i. to homology

$$(A_1 \xrightarrow{P} A_0 \xrightarrow{Q} A_1)^* = (A_1^* \xrightarrow{-Q^*} A_0^* \xrightarrow{P^*} A_1^*)$$

$$\Rightarrow \quad \mathrm{Ext}(A,B) = \mathrm{Tor}(B, A^*)$$

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R, R' : two comm. algebras

B : $R \otimes R'$-bimodule defines a functor

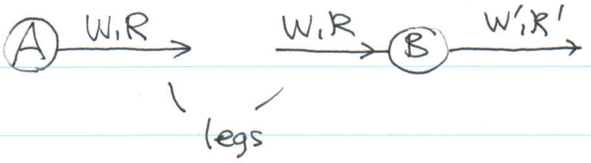
A : R-module $\longmapsto$ $A \otimes_R B$ considered as R'-module

$W \in R$
$W' \in R'$

$B \in MW_{-W+W', R \otimes R'}$

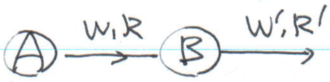
$$\begin{tikzcd} A \arrow[r, mapsto] & A \hat{\otimes}_R B \\ MF_{W,R} \arrow[u, phantom, sloped, "\in"] & MF_{W',R'} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

sl(n)-category
multi-modules を使う.



のときは $\underset{\sim\sim}{-W}$ と約束

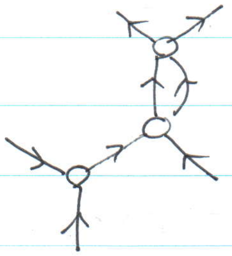
join legs

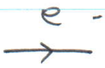


$= A \hat{\otimes}_R B$

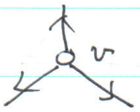
generalization:

decorated graph





··· W, R

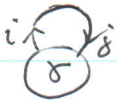


$\hat{v} \in MF^{(-1)^{\text{sign}}}_{\sum W_e}$

incident edges

γ : decorated graph $\longmapsto \hat{\gamma} \in MF_{\sum_{\text{legs } \ell} (-1)^{\text{sign}} W_\ell \,,\, \bigotimes_{\text{legs}} R_\ell}$

s.t. • $\gamma_1 \sqcup \gamma_2$ disjoint $\Rightarrow \widehat{\gamma_1 \sqcup \gamma_2} = \hat{\gamma}_1 \otimes_{\mathbb{Q}} \hat{\gamma}_2$

• 

$\xrightarrow{\text{join}} \widehat{\#_{ij}\gamma} = \hat{\gamma} \big/ (t_{ij} - \mathbb{1})$

and forget about $R_i \otimes R_j$-str.

where $R_i \xrightarrow{t_{ij}} R_j$ isom. s.t. $t_{ij}(W_i) = W_j$.

closed graph ··· cpx over $\mathbb{Q}$

···· can consider homology

Koszul MFs

$p \in R$ $\qquad K(p) = (R_1 \xrightarrow{p} R_0)$ $\qquad$ free resol. of $R/(p)$

$\underset{\underset{\underline{p}\,\in\, R^n}{\shortparallel}}{(p_1,\cdots,p_n) \in R}$ $\qquad K(\underline{p}) = \bigotimes_{i=1}^{n}{}_R\, K(p_i)$

regular sequence : p_i is not a zero divisor in $R/(p_1,\cdots,p_{i-1})$

($\Rightarrow$ free res. of $R/(p_1,\cdots,p_n)$

$$K(\underline{p}) = \Lambda^* R^n \qquad d = \underline{p}\wedge\cdot$$

NB. $\underset{\underset{\underline{p}}{\shortparallel}}{\begin{pmatrix} p_i \\ p_j \end{pmatrix}} \mapsto \underset{\underset{\underline{p}'}{\shortparallel}}{\begin{pmatrix} p_i \\ p_j+\lambda p_i \end{pmatrix}}$ $\quad\Rightarrow K(\underline{p}) \cong K(\underline{p}')$

$\underline{p} \in R^n$, $\underline{q} \in R^{*n}$

$\underset{\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}}{\shortparallel} \qquad \underset{\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}}{\shortparallel}$

$$\underset{\underset{MF_{\sum_{i=1}^{n} p_i \llcorner q_i}}{\cap}}{K(\underline{p};\underline{q})} = \bigotimes_{i=1}^{n}{}_R\, \underbrace{K(p_i, q_i)} \qquad R_1 \xrightarrow{p_i} R_0 \xrightarrow{q_i} R_1$$

pairing

$$K(\underline{p},\underline{q}) = \Lambda^* R^n \qquad D = \underline{p}\wedge + \iota_{\underline{q}}$$

$$D^2 = \underline{p} \llcorner \underline{q}$$

$$K\begin{pmatrix} \vdots & \vdots \\ p_i & q_i \\ \vdots & \vdots \\ p_j & q_j \\ \vdots & \vdots \end{pmatrix} \cong K\begin{pmatrix} \vdots & \vdots \\ p_i & q_i - \lambda q_j \\ \vdots & \vdots \\ p_j + \lambda p_i & q_j \\ \vdots & \vdots \end{pmatrix}$$

Th (Rasmussen)

If $p_1,\cdots,p_n$: regular seq. and $\sum p_i q_i = \sum p_i q_i'$

$\Rightarrow K(\underline{p},\underline{q}) \cong K(\underline{p},\underline{q}')$